# Chaotic family with smooth Lyapunov dependence

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A smooth dependence of the Lyapunov exponent is proved for a nontrivial family of chaotic maps. The approach that is taken demonstrates the importance of Markov partitions in connection with the thermodynamic analysis for dynamical systems. [S1063-651X(97)01306-8]

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Simple one-dimensional maps are often the key for understanding the behavior of complicated physical systems [1]. This, for example, is the case for the different routes by which systems are driven from simple to chaotic behavior [2]. In many cases simple maps already provide a good model for the physical process [3]. It is widely believed that in the parameter region where chaotic behavior is possible, the Lyapunov exponent [4] shows a nonsmooth dependence on the control parameter. As a typical example of such a behavior we mention the logistic map beyond the accumulation point of period doubling [5]. More recently, similar examples have been discovered in connection with the diffusional behavior [6]. There it has been shown that similar effects for tentlike maps on grids of unit cells lead rather directly to a fractal, *Weierstrass-like*, dependence of the diffusion coefficient $D$ on the family parameter (which, in this case, is the slope of the map). In our contribution we show that the parameter dependence can be of smooth nature if the family is appropriately chosen. The work explains implicitly why the parameter dependence is nonsmooth for differently chosen families. As a consequence of our approach it also emerges that it may not be necessary to simulate chaotic systems by a large number of numerical orbits, once the internal (topological) structure of the map is known. Powerful analytical tools allow a thorough investigation of the system without simulations.

The example for which we apply our analytical approach is the bungalow-tent map. This map is of special importance on its own because of the fact that it is the simplest nontrivial linearization of the quadratic parabola (see Fig. 1). Using appropriate parameters, the nonhyperbolic character of the parabola can be carried over to this linearized model, in spite of the linearization. In our contribution, for a specifically chosen one-parameter family of bungalow-tent maps, the Lyapunov exponents are analytically calculated. A smooth dependence of the Lyapunov exponent on the control parameter is found. The analytical results are compared with numerical orbit simulations that yield identical results, from which, however, no statement of smoothness can be derived.

The bungalow-tent maps constitute a two-parameter family of fully developed maps that have a four-piece linearly increasing graph, with a symmetry along a vertical line through $x=0.5$. A two-parameter representation is obtained from the coordinates that fix the location of the right corner point $P$ of the map. The general map then obtains the description

$$f(x):=\begin{cases}\dfrac{b}{1-a}x & \text{for } 0\leqslant x<1-a\\[2mm] \dfrac{b-1}{a-1/2}x+\dfrac{b-1}{2a-1} & \text{for } 1-a\leqslant x<0.5,\\[2mm] \text{symmetric for } 0.5<x\leqslant 1,\end{cases} \tag{1}$$

where $1>a\geqslant 0.5$ and $b/a-0.5>1$ to ensure ergodicity (see Fig. 1). In an earlier paper [7] on the behavior of the bungalow tent map, a one-parameter family was considered by pinning the corner point $P$ on the vertical line $l=(a=0.75,y)$. The dependence of the Lyapunov exponent from the parameter $y$ was shown to be characterized by several phase transition effects in the form of discontinuities of the Lyapunov dependence $\lambda(y)$. Because the size of the Lyapunov exponent is one measure of the unpredictability of the motion, this means that the unpredictability itself varies

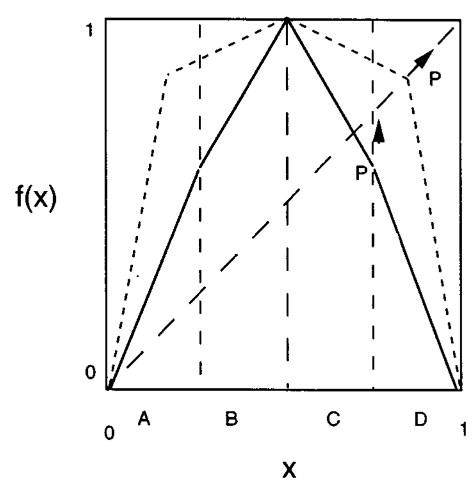


FIG. 1. Graph of the bungalow-tent map. The corner point $P$ determines the structure of the two-parameter family. Two families of bungalow-tent maps are indicated by the direction in which the corner point is moved when the control parameter is changed.

 

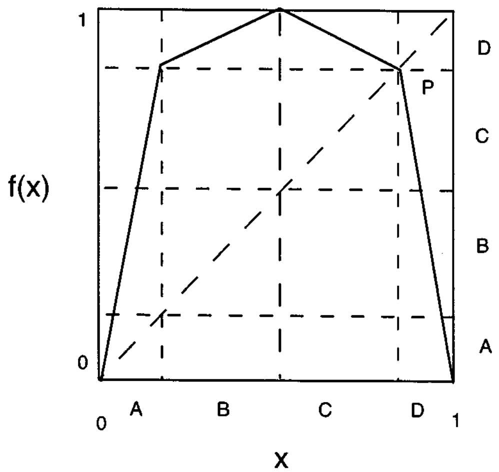


FIG. 2. Markov partition that remains topologically invariant in our family of bungalow-tent maps. A symbolic description is indicated.

in a rather erratic way with the control parameter. For applications (e.g., for chaos control [8]), a smooth dependence instead may be desirable. In this work we chose a different family by varying point $P$ along the diagonal. For this choice, smoothness can be achieved, as we will show below. Both families are displayed in Fig. 1 for convenience. For our choice of the family, the equation of the map is given by

$$f(x):=\begin{cases} \dfrac{a}{1-a}x & \text{for } 0\leqslant x<1-a \\ \dfrac{a-1}{a-1/2}x+\dfrac{a-1}{2a-1} & \text{for } 1-a\leqslant x<0.5, \\ \text{symmetric for } 0.5<x\leqslant 1. \end{cases} \tag{2}$$

For all values of the control parameter $a$, the corner point $P$ is a fixed point. This fixed point is unstable since for all parameters of the family the point is embedded in an unstable two-cycle orbit (despite the fact that the map is not differentiable at $P$). We can see this by inspection of the symbolic partition of the phase space of the map (see Fig. 2). The partition is Markovian [9] for all values of the parameter in our family. This means that under the iteration of the map, the borders of the partition are mapped onto old borders again, which guarantees that the properties of higher-order iterations can be extracted from the partition of the first iteration in a simple way. In the partition, the different regions can be described by symbols, that allow a description in terms of symbolic dynamics [10]. The transfer matrix then describes the transitions among the symbols $A$, $B$, $C$, and $D$ upon the iteration of the map. In this way, the information on the structure of the existing periodic orbits is reflected in the transfer matrix [11]. Of special importance is the fact that in our family the topological structure of the transfer matrix is the same for all values of the control parameter $a$. This is reflected in the simple form of the transfer matrix that is valid for the whole family

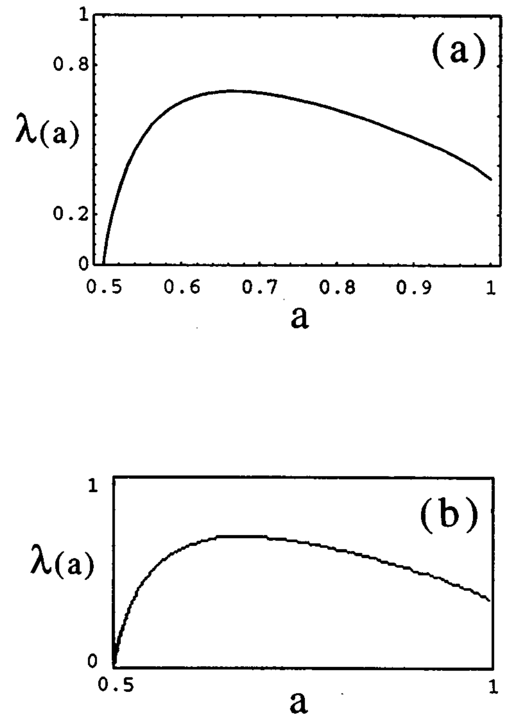


FIG. 3. Dependence of the Lyapunov exponent on the parameter $a$. In our family, the dependence of the Lyapunov exponent on the parameter $a$ is smooth. (a) Analytic results and (b) results from simulations.

$$T=\begin{pmatrix} \dfrac{1-a}{a} & \dfrac{1-a}{a} & \dfrac{1-a}{a} & 0 \\ 0 & 0 & 0 & \dfrac{a-0.5}{1-a} \\ 0 & 0 & 0 & \dfrac{a-0.5}{1-a} \\ \dfrac{1-a}{a} & \dfrac{1-a}{a} & \dfrac{1-a}{a} & 0 \end{pmatrix}. \tag{3}$$

Our analytic solution of the Lyapunov exponent dependence uses this transfer matrix by combining it with thermodynamic formalism methods. In the thermodynamic approach, the elements of the transfer matrix are raised to the power of the (inverse) "temperature" $\beta$, which yields a generalized transfer matrix $T(\beta)$. From this matrix, the (Helmholtz) free energy of the system $F(\beta)$ is obtained as the logarithm of the largest eigenvalue [12]. From the characteristic equation

$$x^4-2\left(\frac{1-a}{a}\right)^{\beta}\left(\frac{a-0.5}{1-a}\right)^{\beta}x^2-\left(\frac{1-a}{a}\right)^{\beta}x^3=0, \tag{4}$$

the largest eigenvalue is obtained as

$$\mu(\beta)=0.5[0.25(r)^{2\beta}+2(r)^{\beta}(z)^{\beta}]^{1/2}+0.5\{[0.25(r)^{2\beta}+2(r)^{\beta}(z)^{\beta}]^{1/2}(r)^{\beta}+0.5(r)^{2\beta}+2(r)^{\beta}(t/s)^{\beta}\}^{1/2}+0.25(r)^{\beta}, \tag{5}$$

where $r=-1+1/a$, $s=1-a$, $t=-0.5+a$, and $z=t/s$. The free energy then emerges as

$$F(a,\beta)=\ln[\mu(\beta)], \tag{6}$$

from which the Lyapunov exponent is obtained via the formula [13]

$$\lambda(a)=-\left.\frac{\partial F(a,\beta)}{\partial\beta}\right|_{\beta=1}. \tag{7}$$

This formula can be derived either directly from a monovariate thermodynamic formalism [13] or via the generalized approach [14]. As general references for the thermodynamic approach to dynamical systems we would like to mention Refs. [12–14]. For our family we obtain the result

$$\begin{aligned}\lambda(a)=&-0.25r\ln(r)-\{0.25[0.5(r)^2\ln(r)+u\ln(r)\\&+u\ln(t/s)]\}/[0.25(r)^2+u]^{1/2}-[0.25([0.25(r)^2\\&+u]^{1/2}r\ln(r)+(r)^2\ln(r)+u\ln(r)+u\ln(z)\\&+\{0.5r[0.5(r)^2\ln(r)+u\ln(r)+u\ln(z)]\}/[0.25(r)^2\\&+u]^{1/2})]/\{[0.25(r)^2+u]^{1/2}r+0.5(r)^2+u\}^{1/2},\end{aligned} \tag{8}$$

where $u=2rz$. The result describes a smooth function of $a$, which is also reflected in the plot shown in Fig. 3. A direct simulation yields the identical dependence. We emphasize that the Lyapunov exponent could also have been calculated by using the invariant measure of the map. From our approach, the invariant measure can be calculated analytically from the eigenvector associated with the leading eigenvalue $\rho=1$ at the temperature $\beta=1$. We illustrate this fact in the plot shown in Fig. 4, where we compare the invariant density calculated from direct simulation and from the eigenvector method. Even more information can be extracted via the thermodynamic approach. Also the different spectra of scaling indices can directly be evaluated from the free energy. In the spectrum of length scales $S(\varepsilon)$ for our family we observe a cutoff of the entropy function at nonzero entropy, on one side of the entropy function (a ''stopping point''). Whether there is a phase transition interpretation [15] of this point still is unclear. Summarizing, we reemphasize the fact that in our example the partition and therefore the topological properties of the map are preserved throughout the family. This is the deeper reason for the smooth dependence of the Lyapunov exponent on the control parameter. While the topological structure remains, the metric and the probabilistic characteristics of the map undergo changes that, themselves, can be related to phase transition phenomena. For $a\to 1$, we detect a behavior of the right piecewise constant natural measure as

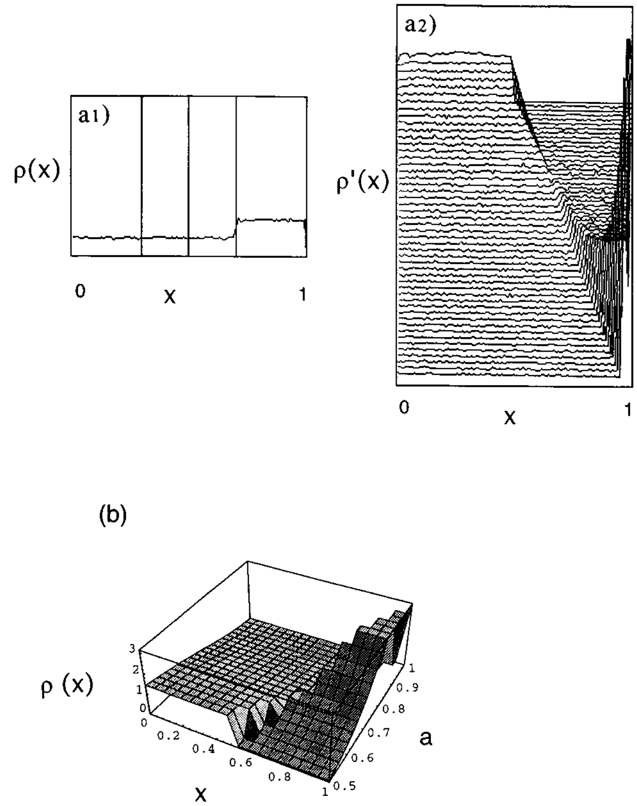


FIG. 4. Natural measure of the map. (a) From simulations [in (a1)] we show the invariant measure for a specific value of the control parameter; in (a2) we plot the shifted density $\rho_a(x)'=\rho_a(x)-ac$, $c>0$, for the whole family. (b) Results from the eigenvector associated with the largest eigenvalue $\mu=1$ at $\beta=1$.

$$\rho(a)\sim|a-1|^{\gamma}, \tag{9}$$

where $\gamma=-1$, so that $\gamma$ can be interpreted as a critical exponent [16].

In conclusion, in the present work we were able to present an explanation for an observed smooth behavior of the Lyapunov dependence for a nontrivial family of chaotic maps. Implicitly, the argument also gives insight into how nonsmooth families emerge. The usage of the thermodynamical formalism for the analytical investigation of chaotic maps has been outlined, where the obtained nontrivial results clearly demonstrate the power of this tool.

[1] M. J. Feigenbaum, J. Stat. Phys. **21**, 669 (1979); Commun. Math. Phys. **77**, 65 (1980); A. Csordás, G. Györgyi, P. Széphalusy, and T. Tél, CHAOS **3**, 31 (1993).
[2] D. Ruelle, *Elements of Differentiable Dynamics* (Academic, San Diego, 1989).
[3] C. Sparrow, *The Lorenz Equations* (Springer, New York, 1980).
[4] R. Stoop and P. F. Meier, J. Opt. Soc. Am. B **5**, 1037 (1988).
[5] See, e.g., P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval* (Birkhäuser, Boston, 1980); W.-H. Steeb, *Chaos und Quantenchaos in Dynamischen Systemen* (BI Wissenschaftliche Verlagsgesellschaft, Mannheim, 1994).
[6] R. Klages and J. R. Dorfman, Phys. Rev. Lett. **74**, 387 (1995).
[7] R. Cluiving, H. W. Capel, and R. A. Pasmanter, Physica A **164**, 593 (1990).
[8] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196

(1990); T. Shinbrot, C. Grebogi, E. Ott, and J. A. Yorke, Nature **363**, 411 (1993); Ying-Cheng Lai and Celso Grebogi, Phys. Rev. E **47**, 2357 (1993).

[9] Y. A. Sinai, *Topics in Ergodic Theory* (Princeton University Press, Princeton, 1994).

[10] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer, Berlin, 1982).

[11] Ch. Beck and F. Schlögel, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, 1985); R. Stoop (unpublished).

[12] G. Radons and R. Stoop, J. Stat. Phys. **82**, 1063 (1996).

[13] M. Kohmoto, Phys. Rev. A **37**, 1345 (1988); J. Peinke, J. Parisi, O. E. Roessler, and R. Stoop, *Encounter with Chaos* (Springer, Berlin, 1992).

[14] T. Tél, *Transient Chaos* in *Directions in Chaos*, edited by Hao Bai-Lin (World Scientific, Singapore, 1990), Vol. 3.

[15] D. Katzen and I. Procaccia, Phys. Rev. Lett. **58**, 1169 (1987), and references therein; E. Ott, C. Grebogi, and J. A. Yorke, Phys. Lett. A **135**, 343 (1989).

[16] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995).